Studies in Geometry

Christopher Arthur

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This paper discusses the following ideas from geometry: tesselators, tubes, regular polyhedral mappings, cubic splines and some special transformations of the unit circle. The reader should have basic knowledge of differential geometry for the discussion of tubes.

1 Tesselators

A Tesselation is commonly known as a tiling of a the plane using a single, primative geometric shape. If the shape is copied and transated, the copies interlock to cover the entire plane without any overlapping. The idea of a tesselation is expanded by Kepler to allow not just single shapes, but tilings by aggregate patterns of regular and star polygons. There is in either case a finite pattern. A **tesselator** is a relation that enumerates each of the copies. It serves to organize a tesselation for the purposes of projection onto manifolds.

We introduce two kinds of tesselators.

1.1 Toriodal Tesselators

$T_T : \mathbf{R}^2 \longrightarrow \mathbf{Z}^2$ is a surjective map that essentially assigns a unique pair of integers (i, j) to each instance of the pattern. For the purpose of building aggregate tesselations, the pattern is topologically equivalent to a torus, so that we can imagine it also as a rectangle having the top meeting the bottom and the left meeting the right.

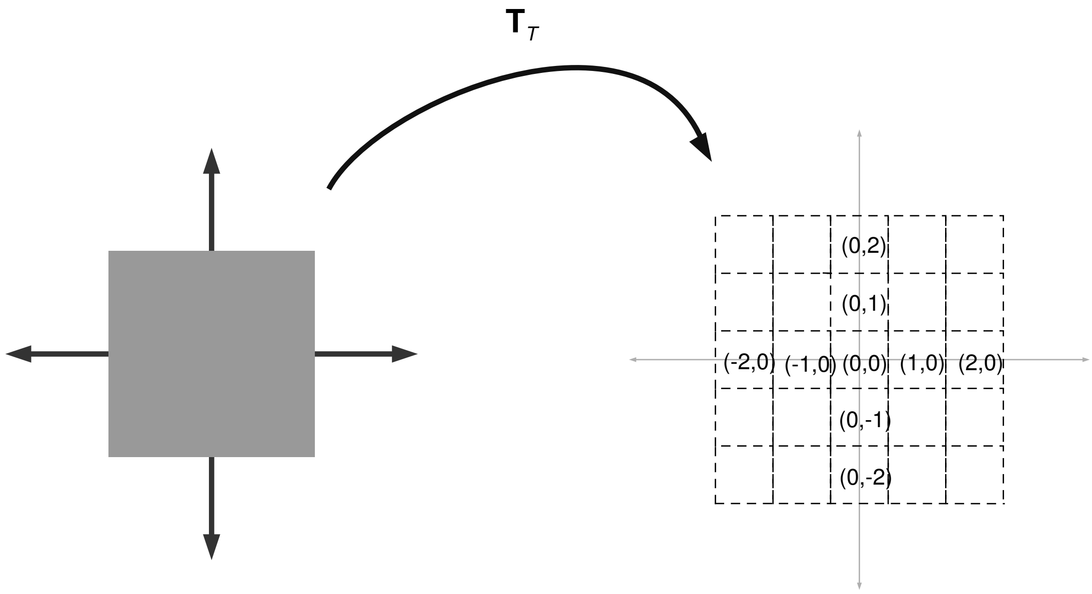

Figure 1: Enumeration by toroidal tesselator

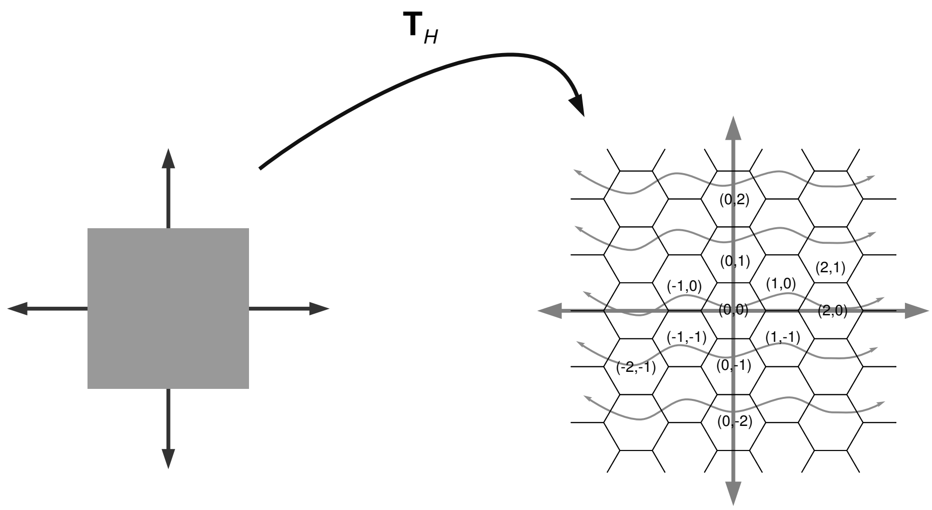

Figure 2: Enumeration by hexagonal tesselator

1.2 Hexagonal Tesselators

$T_H : \mathbf{R}^2 \longrightarrow \mathbf{Z}^2$ is similiar to T_T, only that the patterns are enumerated in staggered rows due to the nature of how hexagons interlock.

2 Tubes

A **tube** $\mathfrak{T}(C, \mathbf{s})$ in $\mathbf{R}^3$ is a **cross-section** C extruded along a path $\mathbf{s}$. The cross-section $c : [0, 1] \longrightarrow C \subset \mathbf{R}^2$ is a closed curve with no intersections (an injective map). The path $\mathbf{s} : A \subset \mathbf{R} \longrightarrow \mathbf{R}^3$ is a continuously differentiable C^2 curve.

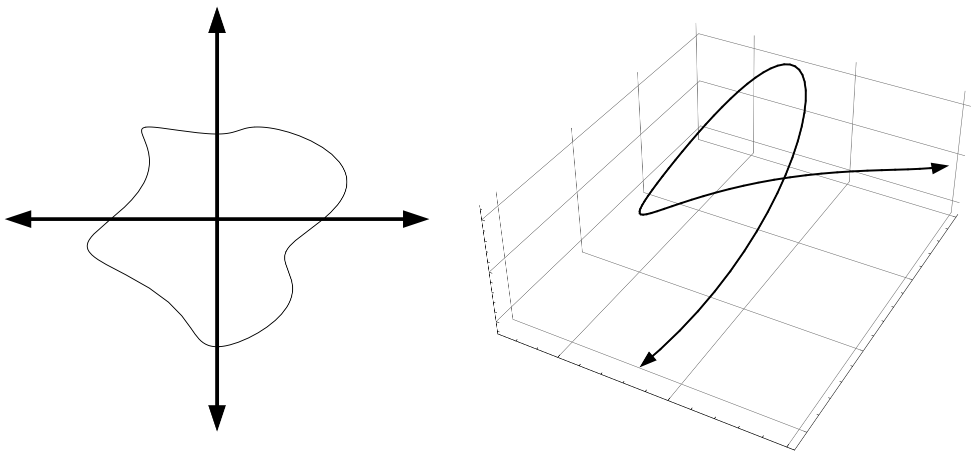

Figure 3: Cross-section and path

2.1 Construction

To construct a tube from its path and cross-section, we project the cross-section at each point along the path. We first derive a basis for the plane into which we make the projection, then define a mapping from the space of the cross-section. Let $t \in A$ be arbitrary, $p = \mathbf{s}(t)$ and $v = \mathbf{s}'(t)$ so that $v_p \in \mathbf{R}^3_p$. For the cross-section, let $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{R}^2$ be a basis. At point p we have a plane normal to v with basis $(\hat{\mathbf{e}}^1)_p, (\hat{\mathbf{e}}^2)_p \in \mathbf{R}^3_p$. For each $a \in C$, we have

$a = x\mathbf{e}_1 + y\mathbf{e}_2$. The projection $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ that maps the cross-section is defined as

$$T(a) = \begin{bmatrix} \hat{\mathbf{e}}_1^1 & \hat{\mathbf{e}}_2^1 \\ \hat{\mathbf{e}}_1^2 & \hat{\mathbf{e}}_2^2 \\ \hat{\mathbf{e}}_1^3 & \hat{\mathbf{e}}_2^3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + p$$

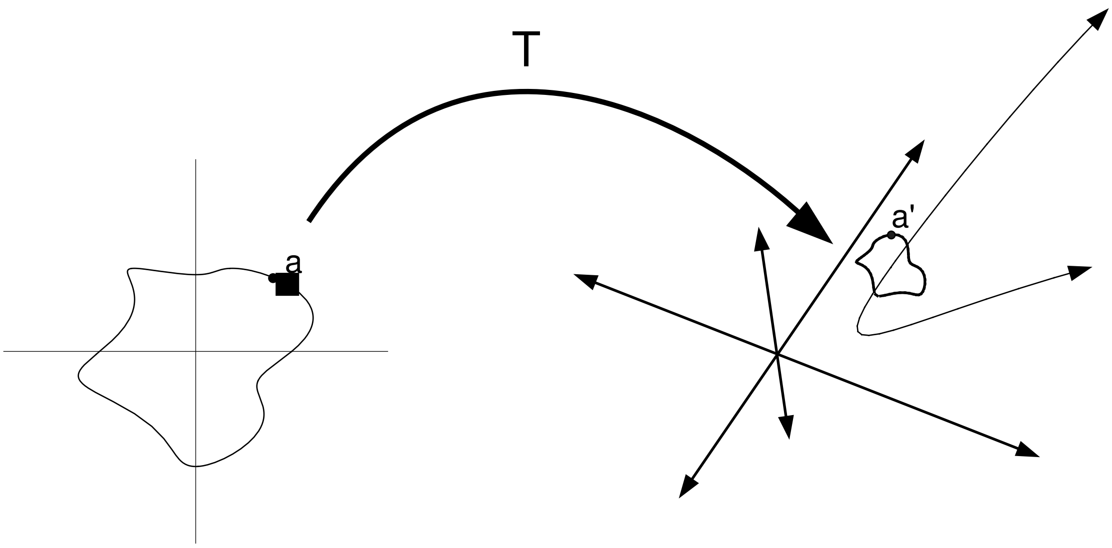

Figure 4: Mapping cross-section onto path, $(a' = T(a))$

Now, as t is arbitrary, this mapping can be repeated for all t. As an example, consider the usual basis for the cross-section space, and for the planes, we derive an expression for the basis at a point by parametrizing everything in terms of t:

$$\mathbf{n}(t) = \frac{\mathbf{s}'(t)}{\|\mathbf{s}'(t)\|}$$

$$\hat{\mathbf{e}}_1(t) = \frac{\mathbf{n}'(t)}{\|\mathbf{n}'(t)\|}$$

$$\hat{\mathbf{e}}_2(t) = \mathbf{n}(t) \times \hat{\mathbf{e}}_1(t)$$

We obtain a coordinate system $f : A \times [0,1] \longrightarrow \mathbf{R}^3$ for the entire tube surface.

$$f(t,u) = (c_1(u)\hat{\mathbf{e}}_1(t) + c_2(u)\hat{\mathbf{e}}_2(t) + \mathbf{s}(t)$$

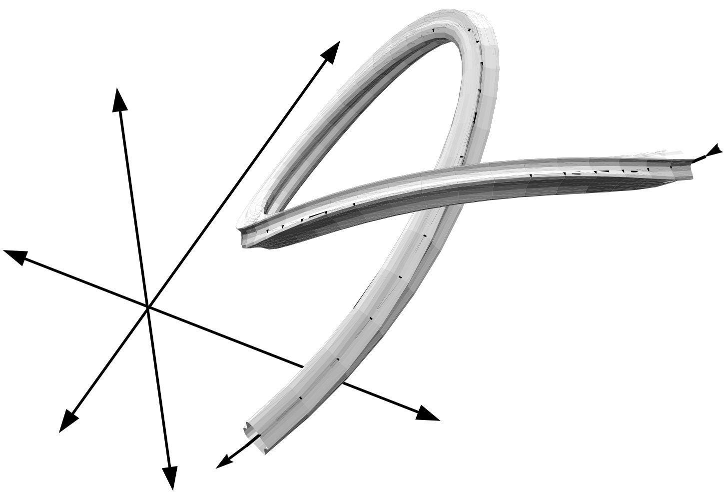

Figure 5: Complete tube $\mathfrak{T}(C, \mathbf{s})$

3 Regular Polyhedral Mappings

A **polyhedral map** $P = \{P_i\}_{i=1}^n$ is a set of transformations $P_i : V \longrightarrow W_i$ defined on the a bounded space $V = A \times [0, \infty)$ where A is a polygon in $\mathbf{R}^2$. A polyhedral map is *regular* if:

1. A is a face of a regular polyhedron.
2. P maps to the volumes above each face on the polyhedron.
3. $W_i \cap W_j = \emptyset$ for all i, j.

For example, for a cube we have $P = \{P_i\}_{i=1}^6$ for the six sides with $A = [-0.5, 0.5] \times [-0.5, 0.5]$.

Note that P_i is essentially a rotation followed by a translation.

One application of polyhedral maps is in the construction of symmetric solids and surfaces in which the symmetry is guided by underlying polyhedra. By carefully choosing the boundary of a subset of V, $P(V)$ becomes a surface with continuous transitions from one face to the next.

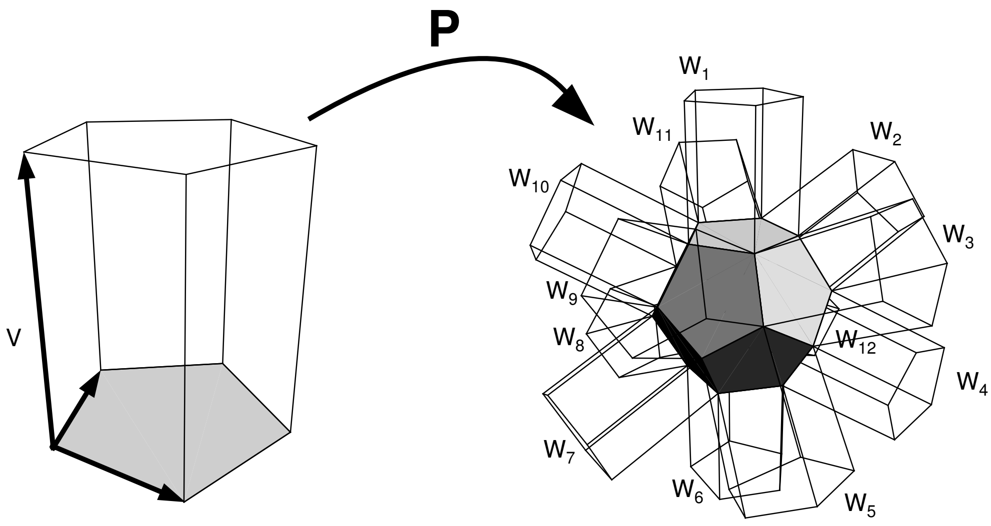

Figure 6: A dodecahedral map

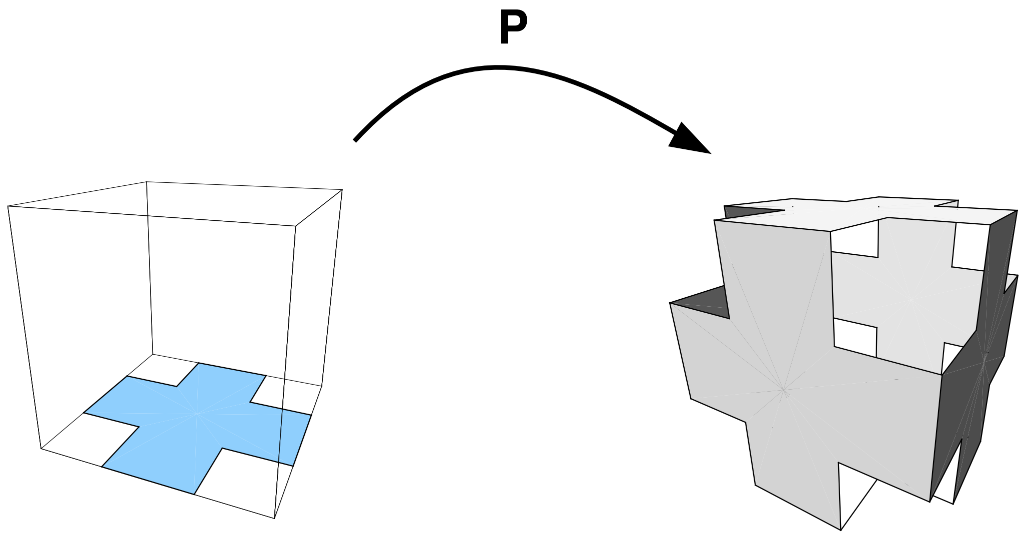

Figure 7: A cube mapping of a cross

4 Determing Cubic Spline Functions

Arising frequently in geometry is need for a smooth curve between two endpoints, with the added requirement that the direction of the curve at its endpoints can be specified. Cubic splines provide a straightforward solution and a simple algebraic means to derive a continously differentiable function for the curve.

Let X be the set of all cubic polynomials having real coefficients. A **cubic spline seqment** in $\mathbf{R}^n$ is a function $\mathbf{s} : A \subset \mathbf{R} \longrightarrow \mathbf{R}^n$ such that $\mathbf{s} : t \longmapsto (f_1(t), f_2(t), ..., f_n(t))$ where $\{f_i\}_{i=1}^n \in X$. For isolated segments, it is convenient to consider $A = [0, 1]$. A cubic spline segment is a smooth curve that can be completely determined by the position and heading at its endpoints. The headings are essentially the derivatives of the curve at the endpoints (more properly, they are the limits of the derivatives).

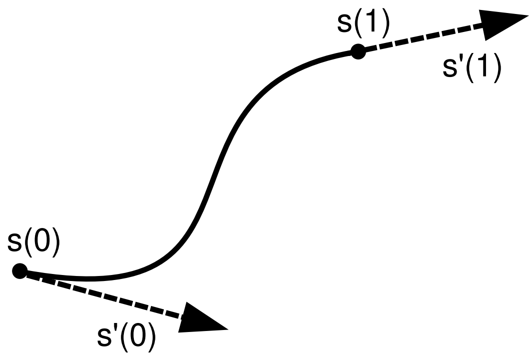

Figure 8: Heading and position of endpoints

If the endpoints and headings are known, we can determine each f_i by solving a system of linear equations. Consider first the case of a spline segment in $\mathbf{R}$. Then $\mathbf{s}(t) = f(t)$. Let $f(t) = a_0 + a_1 t + a_2^2 + a_3 t^3$. By differentiation of f and substitution, we have

$$\begin{aligned} \mathbf{s}(0) &= a_0 \\ \mathbf{s}(1) &= a_0 + a_1 + a_2 + a_3 \\ \mathbf{s}'(0) &= a_1 \\ \mathbf{s}'(1) &= a_1 + 2a_2 + 3a_3 \end{aligned}$$

which yields the following solutions:

$$
\begin{aligned}
a_0 &= \mathbf{s}(0)\\
a_1 &= \mathbf{s}'(0)\\
a_2 &= -3\mathbf{s}(0) + 3\mathbf{s}(1) - 2\mathbf{s}'(0) - \mathbf{s}'(1)\\
a_3 &= 2\mathbf{s}(0) - 2\mathbf{s}(1) + \mathbf{s}'(0) + \mathbf{s}'(1)
\end{aligned}
$$

For the case in $\mathbf{R}^n$, we have n systems of equations all with similar solutions. We note that $\pi^i \circ \mathbf{s} = f_i$, and generate the systems accordingly.

A **cubic spline** in $\mathbf{R}^n$ may be expressed as a piecewise function composed of a series of segments, typically linked together to form a continuously differentiable C^1 curve.

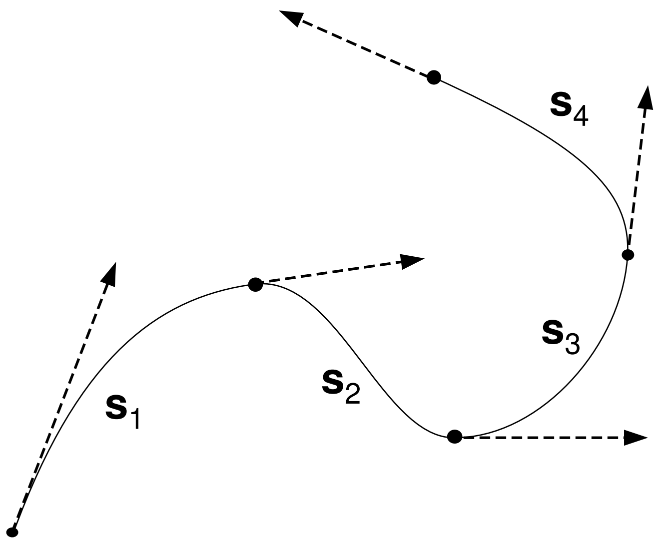

Figure 9: Segments chained together

If S is a cubic spline having two segments $\mathbf{s}_1$ and $\mathbf{s}_2$, then we could have, for example, the domains of $\mathbf{s}_1$ as $[0,1]$ and $\mathbf{s}_2$ as [1,2], and the chaining condition that $\mathbf{s}_1(1) = \mathbf{s}_2(1)$ and $\mathbf{s}_1'(1) = \mathbf{s}_2'(1)$.

5 Special Transformations of the Unit Circle

Let $\mathbf{s}$ be a parameterization of the unit circle A.

$$\mathbf{s} : [0,1] \longrightarrow A \subset \mathbf{R}^2$$

such that

$$\mathbf{s} : t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

A transformation $T_r : A \longrightarrow B \subset \mathbf{R}^2$ is a **radial modulator** if there exists a function

$$r : [0,1] \longrightarrow \mathbf{R}$$

such that

$$T_r(A) = \{r(t)\mathbf{s}(t) | t \in [0,1]\}$$

T_r transforms the circle by changing its radius at each point. We show that T_r is well-defined. Let $x, y \in A$ such that $x = y$. Since $\mathbf{s}$ is injective, there exist $a, b \in [0,1]$ where $a = \mathbf{s}^{-1}(x)$ and $b = \mathbf{s}^{-1}(y)$. Now $x = y$, so $\mathbf{s}(a) = \mathbf{s}(b) \Rightarrow r(a)\mathbf{s}(a) = r(b)\mathbf{s}(b)$. Since, $T_r \circ \mathbf{s}(t) = r(t)\mathbf{s}(t) \forall t$, $T_r \circ \mathbf{s}(a) = T_r \circ \mathbf{s}(b)$ and finally $T_r(x) = T_r(y)$.□

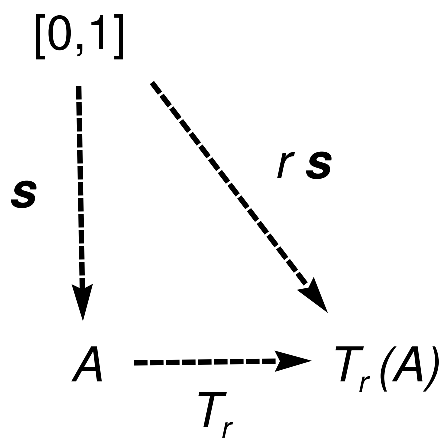

Figure 10: Diagram of radial modulator

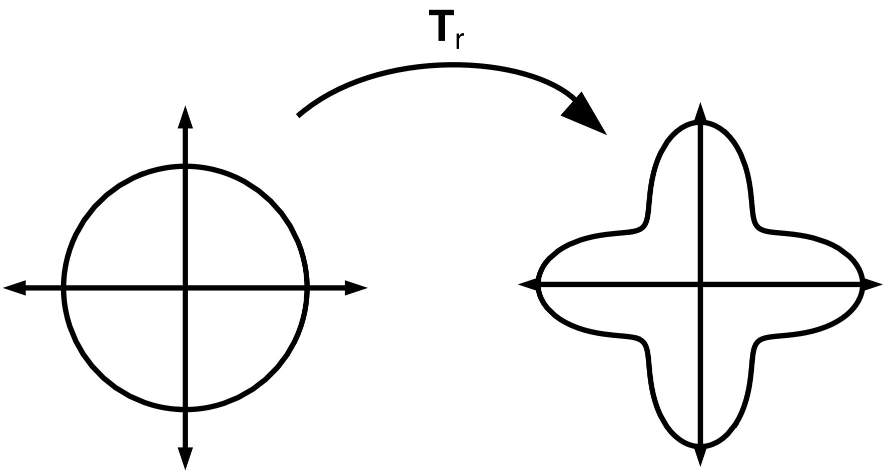

Figure 11: $r(t) = \frac{1}{3}\cos 8\pi t + 1$

A transformation T_t is a **toroidal modulator** if there exists a function

$$\rho : [0,1] \longrightarrow \mathbf{R}^2$$

such that $\rho : t \mapsto (R_1 - R_2 \cos 2\pi nt, R_2 \sin 2\pi nt)$ and $T_t(A) = \{(\rho_1 \mathbf{s}_1, \rho_1 \mathbf{s}_2, \rho_2 \mathbf{s}_3) | t \in [0,1]\}$ with $R_1, R_2 \in \mathbf{R}$ and $n \in \mathbf{Z}^+$. For convenience we define a function

$$G : [0,1] \longrightarrow \mathbf{R}^3$$

such that $G = (\rho_1 \mathbf{s}_1, \rho_1 \mathbf{s}_2, \rho_2 \mathbf{s}_3)$. The proof that T_t is well-defined proceeds as before, noting that for $x, y \in A$, $x = y \Rightarrow G(\mathbf{s}^{-1}(x)) = G(\mathbf{s}^{-1}(y)) \Rightarrow T(x) = T(y)$.

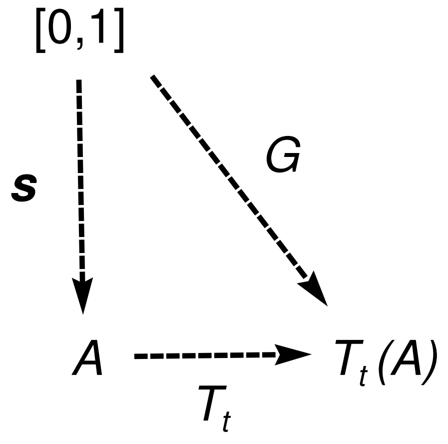

Figure 12: Diagram of toroidal modulator

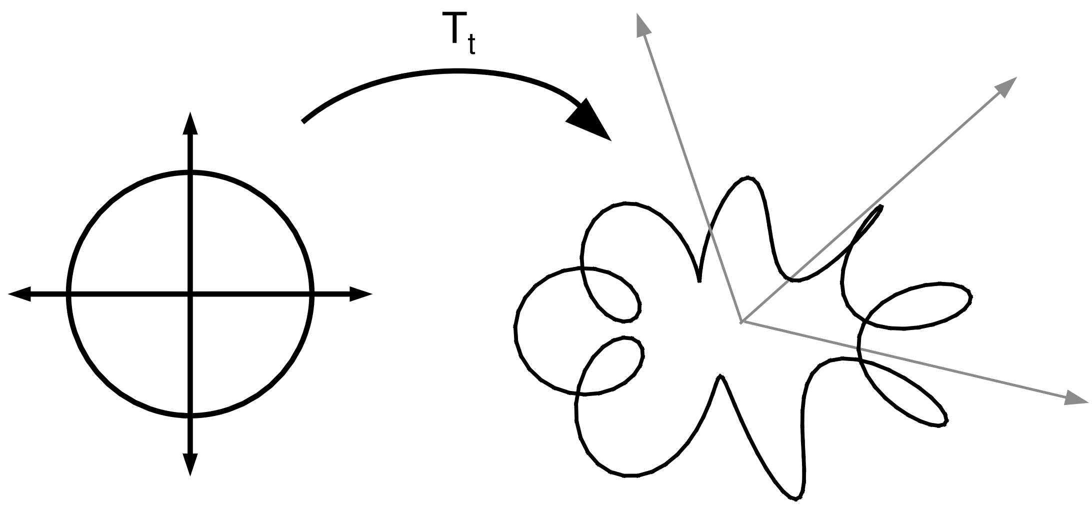

Figure 13: $\rho(t) = (1 - \frac{1}{3}\cos 2\pi 8t, \frac{1}{3}\sin 2\pi 8t)$

References

[1] Spivak, Michael.(1965). *Calculus on Manifolds.*

[2] Eric W. Weisstein. "Transformation." From *MathWorld. http://mathworld.wolfram.com/Transformation.html*